Article

Numerical Fractal Analysis of Exceptional Sets in the Lehner Expansion

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Abstract

This paper examines how the average value of the sequence b_n in the Lehner expansion of a real number x influences its box dimension. Our primary objective is to analyze how variations in the average of b_n impact the box dimension, which serves as a measure of the complexity of the sequence. Using the box-counting method, we numerically estimate the box dimension and explore its relationship with the fractal nature of Lehner expansions.

Keywords: Regular continued fraction, Lehner expansion, semi-regular continued fraction,fractal dimension, box dimension.

I. Introduction

Continued fractions [1]–[4] play important role in number theory,this way of writing numbers is very useful in number theory because it helps us understand how well we can approximate real numbers using fractions.A semi-regular continued fraction [7] is a special type of continued fraction that extends classical regular continued fractions by allowing a broader set of partial quotients while keeping important mathematical properties.One such semi-regular expansion is the Lehner continued fraction, which has been investigated for its unique convergence behavior and number-theoretic significance (Lehner, 1949).
Any irrational number $x \in [1, 2]$ has a unique Lehner expansion of the form

$$b_0 + \cfrac{\sigma_1}{b_1 + \cfrac{\sigma_2}{b_2 + \cdots + \cfrac{\sigma_n}{b_n + \dots}}} = [b_0; \sigma_1/b_1, \sigma_2/b_2, \dots, \sigma_n/b_n, \dots]$$

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(1)

where (b_i, σ_{i+1}) equals (1,1) or (2,-1).We call these continued fractions Lehner fractions or Lehner expansions. Every rational number has two different finite Lehner expansions.
Lehner expansions can be found using this map $L : [1, 2) \to [1, 2)$, which is defined as follows.

$$Lx := \begin{cases} \frac{1}{2-x}, & 1 \leq x < \frac{3}{2}, \\ \frac{1}{x-1}, & \frac{3}{2} \leq x < 2. \end{cases}$$

Notice that in this expansion for $x \in [1, 2)$ one has

$$(b_i, \sigma_{i+1}) = \begin{cases} (1, 1), & \text{if } L^i(x) \in \left[\frac{3}{2}, 2\right), \\ (2, -1), & \text{if } L^i(x) \in \left[1, \frac{3}{2}\right). \end{cases}$$

Lehner expansions are a type of semi-regular continued fraction. A semi-regular continued fraction can be either a finite or an infinite fraction.

The study of exceptional sets in continued fractions has been a focus of recent research, exploring their fractal properties and Hausdorff dimensions. Fang et al. [8] determined the Hausdorff dimension of a set related to the growth rate of continued fraction coefficients. Kazin and Kadyrov [5] extended Good's work on fractal geometry in continued fractions, establishing new bounds on Hausdorff dimensions of level sets formed by restricting partial quotients. Bakhtawar et al. [9] calculated the Hausdorff dimension of a set defined by conditions on ratios of consecutive continued fraction coefficients, contributing to the metrical theory of continued fractions. While not directly addressing continued fractions, Parsell and Wooley [10] investigated exceptional sets for Diophantine inequalities, showing that under certain conditions, the measure of the exceptional set in an interval is bounded. These studies collectively advance our understanding of exceptional sets in number theory and their geometric properties fractal properties of these sets, particularly their Hausdorff and box dimensions, have been the subject of extensive research [11]. For regular continued fractions, the dimension of sets defined by constraints on their partial quotients has been thoroughly examined [4], [12]. However, for semi-regular expansions such as the Lehner continued fraction, a comprehensive understanding of these exceptional sets remains incomplete.

Fractal dimension measures how completely a fractal fills space as one zooms in on finer scales. Unlike traditional Euclidean dimensions, which take integer values (e.g., a line has dimension 1, a plane has dimension 2), fractal dimensions can be non-integer, reflecting the complexity and self-similarity of fractal structures. It quantifies how detail in a pattern changes with the scale at which it is measured, making it useful for characterizing irregular shapes in nature, such as coastlines, clouds, and turbulent flows. The Hausdorff dimension and box dimension are both types of fractal dimensions. For more information on how these various notions of dimension are related, we refer to [6]. In this paper, we focus only on the box dimension.The box dimension of set S is defined as

$$\dim_B(S) = \lim_{\delta \to 0} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the number of boxes size δ required to cover set S.

If this limit exists. This dimension captures how the number of covering elements scales with their size and provides a practical way to estimate fractal complexity.

Theorem [7,theorem4] *For almost all real numbers* $x \in (1, 2)$, *we have that their Lehner expansions*

$$x = [b_0; \sigma_1/b_1, \sigma_2/b_2, \cdots, \sigma_n/b_n, \cdots]$$

$$\lim_{n \to \infty} \frac{b_1 + b_2 + \cdots + b_n}{n} = 2.$$

In this work, we focus on the set of numbers for which the sequence of partial quotients b_n in the Lehner expansion exhibits an anomalous growth pattern, specifically cases where the long-term average deviates from its expected limit. Such deviations are known to correspond to fractal-like structures, whose complexity can be quantified using box dimension [13].Our objective is to determine how the box dimension of these exceptional sets depends on the asymptotic behavior of b_n, extending results known for regular continued fractions []. We consider those real numbers x for which the above limit is not equal to 2. By the theorem we

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know that this set has Lebesgue measure zero. However, it may have a complex structure from a fractal geometry point of view. To understand, for any $\epsilon > 0$ we define sets

$$S(\epsilon, c) = \left\{ x \in (1, 2) : \lim_{n\to\infty} \frac{b_1 + b_2 + \cdots + b_n}{n} \in (c - \epsilon, c + \epsilon) \right\}.$$

Our research question is to numerically investigate how box dimension of $S(\epsilon, c)$ depends on ϵ. For the definition of box dimension, see the next section.

To achieve this, we employ a computational approach based on binary word representations, adapting established methods from multifractal analysis (Barreira & Schmeling, 2000). By numerically estimating the box dimension for different classes of exceptional sets, we provide new insights into the geometric complexity of Lehner continued fraction expansions. Our findings contribute to the broader understanding of fractal structures in number theory and highlight the rich interplay between continued fractions and dynamical systems.

The structure of the paper is as follows. In Section 2, we introduce the mathematical framework of continued Lehner fractions and review key definitions. Section 3 describes the methodology for computing the box dimension, detailing the binary word-based approach. Section 4 presents numerical results and discusses the implications of our findings. Finally, in Section 5, we summarize our conclusions and suggest directions for future research.

II. Methodology

The box dimension of set S is defined as

$$\dim_B(S) = \lim_{\delta\to 0} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the number of boxes size δ required to cover set S. This dimension characterizes the fractal scaling behavior of the set as the solution δ decreases.

To estimate the box dimension numerically, we analyze the scaling behavior of unique truncated binary words derived from points in a given set. Each point in the set is mapped to a binary expansion with fixed precision by repeatedly multiplying the point by two. If the result is at least one, '1' is appended to the binary string, and one is subtracted from the point; otherwise, '0' is appended. This process is repeated for the desired precision; see Fig.1.

```
Input:
    x - a real number in [0,1)
    precision - number of bits in the binary expansion
Output:
    binary_expansion - a string representing the binary expansion
Initialize binary_expansion as an empty string
For i from 1 to precision do
  x ← 2x
  If x ≥ 1 then:
    Append "1" to binary_expansion
    x ← x − 1
  Else:
    Append "0" to binary_expansion
Return binary_expansion
```

Fig. 1. Algorithm to compute the binary expansion of real numbers

To estimate the complexity of the given set, each binary expansion is truncated to a fixed word length, meaning that only the first few digits of the binary representation are considered. For each chosen word length, the number of unique binary words

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(subsequences of that length) is counted. This process is repeated for multiple word lengths, allowing us to analyze how the number of distinct binary words grows as the word length increases.

Next, the base-2 logarithm of the number of unique binary words is computed for each word length. This step helps to transform the data into a form that reveals scaling properties. The resulting data points, which represent the relationship between word length and the logarithm of the unique word count, are then analyzed using linear regression. Linear regression is used to fit a trend line to the data, which captures the overall pattern of growth.

Once the trend line is obtained, the box dimension of the set is determined by the slope of the regression line. This slope quantifies how the number of unique binary words scales with word length and provides a numerical measure of the set's complexity. A higher slope indicates greater complexity, while a lower slope suggests a more structured or predictable pattern in the binary expansions.

The computational procedure follows these steps: The Set points are first converted to binary expansions of a specified precision. For each word length, the binary expansions are truncated, and the number of unique words is counted. The $\log_2$ of the unique word count is computed and stored. Then a linear regression is performed on the relationship between word length and $\log_2$ count, and the slope of the regression line is returned as the estimated box dimension. The results are visualized through a regression plot that shows the relationship between word length and $\log_2$ of unique word counts, where the slope of the fitted trend line provides an approximation of the box dimension of the underlying fractal set.The following pseudocode Fig.2. summarizes the computational procedure:

Input: Set points, precision p, max word length L

Output: Estimated box dimension

1. Convert each set point to a binary expansion of length p.
2. Initialize an empty list for log_2 counts.
3. For each word length l from 1 to L:
    a. Truncate each binary expansion to the first l bits.
    b. Count the number of unique truncated words.
    c. Compute log_2 of the unique word count and store it.
4. Perform linear regression on (word length, log_2 count) pairs.
5. Return the slope of the regression line as the estimated box dimension.

Fig. 2. Algorithm to numerically compute the box dimension

To carry out the experiments we generated one million points uniformly from the interval [1,2]. The distribution of denominator averages of these numbers are depicted in Fig.3.

III. Results and Discussion

Fig.4 provide numerical results for estimating the Box dimension of $S(\epsilon, c)$ for fixed ϵ=0.01 and c ranging from 1.60 to 1.95.Our numerical investigation of the box dimension of $S(\epsilon, c)$ reveals a clear dependence on c.Using the binary word-based box-counting method, we estimated the box dimension of these exceptional sets. By comparing the log of unique binary word counts to word length using linear regression, we found a slope that shows how $S(\epsilon, c)$ scales in a fractal way.

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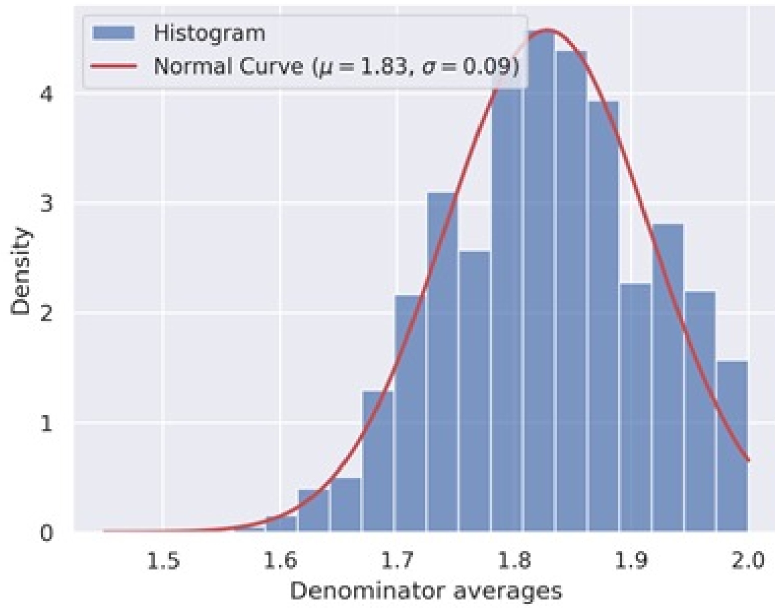

Fig. 3. Histogram plot of relative frequency distribution of average denominators of Lehner expansion

The Fig. 5 suggests that as c increase, the box dimension stabilizes, reinforcing the theoretical expectation that the set of exceptions forms a measure-zero yet structurally complex subset.

Fig.6(a) depicts a fractal structure generated from continued fraction expansions with a restricted digit set { 1, 2, 3, 4 }. The x and y coordinates correspond to values derived from odd- and even-indexed terms of randomly generated continued fraction sequences. The resulting structure reveals an intricate, self-similar distribution within the unit square, illustrating how different digit choices influence the fractal pattern. Fig.6 (b) shows a similar fractal formation, but based on the Lehner expansion, a variant of continued fraction representation defined for numbers in the interval [1,2]. Here, the x and y coordinates are determined by evaluating the odd- and even-indexed Lehner terms as continued fractions. The clustering and density variations within the bounded region reflect the distinctive number-theoretic properties of the Lehner transformation and its role in generating self-similar structures.

IV. Conclusion

In this paper, we investigated the fractal properties of exceptional sets in the Lehner expansion by examining how the average value of the sequence b_n affects the box dimension. By employing numerical fractal analysis, we computed the box dimensions of these exceptional sets using a binary word-based box-counting method. Our findings demonstrate that the box dimension of $S(\epsilon, c)$ exhibits a clear dependence on c, with the box dimension stabilizing as c increases. This aligns with the theoretical expectation that these sets, despite having Lebesgue measure zero, exhibit intricate fractal structures.

Our numerical results provide evidence that the exceptional sets in the Lehner expansion possess a non-trivial fractal nature, reinforcing the idea that continued fraction expansions offer a rich framework for studying complex structures in number theory. The observed self-similar patterns in Fig 6(a) and 6(b) further illustrate how the Lehner expansion differs from regular continued fractions while maintaining its own unique fractal characteristics. The histogram of denominator averages (Fig.3) and the scaling

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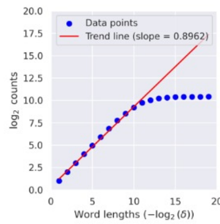
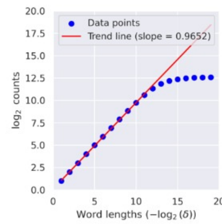
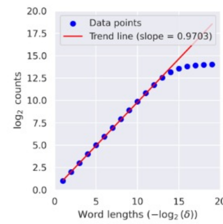


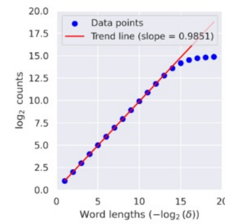
(a) $S(0.01,1.60)$



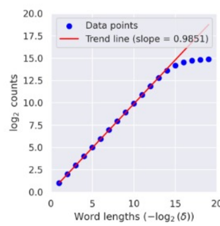
(b) $S(0.01,1.65)$



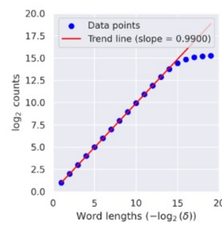
(c) $S(0.01,1.70)$



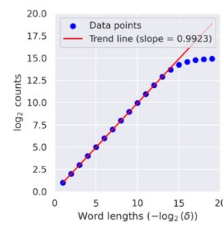
(d) $S(0.01,1.75)$



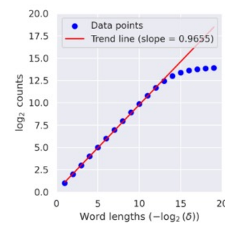
(e) $S(0.01,1.80)$



(f) $S(0.01,1.85)$



(g) $S(0.01,1.90)$



(h) $S(0.01,1.95)$

Fig. 4. Numerical box dimension estimates of $S(\epsilon, c)$ for ϵ=0.01 and varying c.

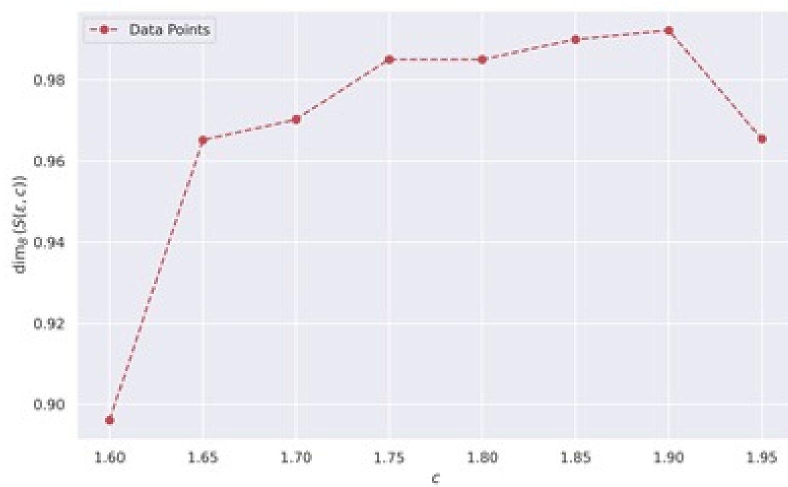


Fig. 5. The graph of box dimensions of $S(\epsilon, c)$ as c changes from 1.60 to 1.95

behavior of box dimensions (Fig.5) suggest that the complexity of these sets is deeply tied to the digit distributions in their continued fraction representations.

Future work could extend this study by exploring different ranges of ϵ and c to further characterize the transition behaviors of fractal dimensions. Additionally, a theoretical analysis of the scaling behavior observed in our numerical experiments could provide deeper insights into the number-theoretic properties of Lehner expansions. Overall, this study contributes to the growing body of

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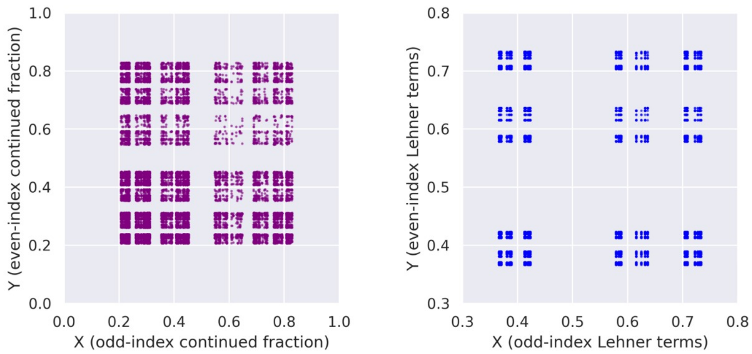

(a) 2D fractal structure of regular continued fractions

(b) 2D fractal structure of Lehner continued fractions

Fig. 6. Two-Dimensional Fractal Structures from Continued Fraction Expansions

research on the fractal geometry of exceptional sets in continued fraction theory, offering new perspectives on their complexity and structure.

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