

## Article

# Varieties of right-commutative algebras defined by the identity $a(aa) = 0$

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## Abstract

An algebra  $R$  is called a *right-commutative* algebra if it satisfies the right-commutative identity

$$(x_1x_2)x_3 - (x_1x_3)x_2 = 0,$$

for all elements  $x_1, x_2, x_3 \in R$ .

This algebra is also well known by another name, for example, *NAP* algebra or a *non-associative permutative* algebra. Varieties of right-commutative algebras contain important subvarieties such as Novikov and bicommutative algebras.

In this paper, we consider two subvarieties  $\mathfrak{M}$  and  $\mathfrak{N}$  of the variety of right-commutative algebras. The subvariety  $\mathfrak{M}$  is defined by the identities  $x_1(x_1x_1) = 0$ ,  $x_1[x_1, x_2] = 0$  and  $x_1(x_2x_1) - x_2(x_1x_1) = 0$ . The subvariety  $\mathfrak{N}$  is defined by the identities  $x_1(x_1x_1) = 0$ ,  $x_1(x_1x_2) - 2x_1(x_2x_1) + x_2(x_1x_1) = 0$  and  $[x_1, x_2]x_1 = 0$ .

We study the free algebras in the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$  of right-commutative algebras. We investigate the  $S_n$ -module structures of the free algebras  $F_n(\mathfrak{M})$  and  $F_n(\mathfrak{N})$  in the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ , respectively. As a consequence, we prove that the subvarieties  $\mathfrak{M}$  and  $\mathfrak{N}$  of the variety of right-commutative algebras satisfy the distributivity conditions. Moreover, we describe all subvarieties of the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ .

**Keywords:** variety of algebras, lattice of subvarieties, free algebra, right-commutative algebra,  $S_n$ -module structure

## I. INTRODUCTION

An algebra  $R$  is called a *right-commutative* algebra if for all elements  $x_1, x_2, x_3 \in R$

$$(x_1x_2)x_3 - (x_1x_3)x_2 = 0. \tag{1}$$

This algebra is also well known by another name, for example, *NAP* algebra or *non-associative permutative* algebra, see [1], [2]. The variety of *NAP* algebras contains several important subvarieties. In particular, the varieties of left-Novikov algebras and bicommutative algebras (or *LR*-algebras) are examples of right-commutative algebras. These varieties have been extensively studied

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from the point of view of free algebras, codimension growth, operads, and polynomial identities. For example, in [3], a free basis was constructed and the codimensions of Novikov algebras were found. Moreover, it was shown that the Novikov operad is not Koszul operad. In [4], the  $S_n$ -module structures of free Novikov algebras were investigated.

LR-algebras first appeared in the work of Dzhumadil'daev and Tulenbaev [5]. In [6], a free basis was constructed, the codimensions were calculated, and the  $S_n$ -module structures of free LR-algebras were fully studied. In 2018, Drensky and Zhakhayev [7] gave an affirmative answer to the Specht problem for LR-algebras.

Let  $\mathfrak{V}$  be a variety of algebras. The set of all subvarieties of the variety  $\mathfrak{V}$  of algebras form a lattice  $L(\mathfrak{V})$  with respect to the inclusion order. The lattice  $L(\mathfrak{V})$  is distributive if for all subvarieties  $\mathfrak{V}_1, \mathfrak{V}_2$ , and  $\mathfrak{V}_3 \subseteq \mathfrak{V}$  the following identity holds

$$(\mathfrak{V}_1 \vee \mathfrak{V}_2) \wedge \mathfrak{V}_3 = (\mathfrak{V}_1 \wedge \mathfrak{V}_3) \vee (\mathfrak{V}_2 \wedge \mathfrak{V}_3).$$

In 1976, Bokut [8] formulated the following question: to describe, in terms of identities, the varieties of algebras whose lattice of subvarieties is distributive. In 1976, Ananin and Kemer [2] proved that the lattice of subvarieties of the variety  $\mathcal{A}$  of associative algebras over a field  $K$  of characteristic zero is distributive if and only if the  $T$ -ideal of identities of  $\mathcal{A}$  contains an identity of the form

$$\alpha[x, y]y + \beta y[x, y] = 0,$$

where  $\alpha, \beta \in K$  and  $(\alpha, \beta) \neq (0, 0)$ .

In [10] and [11], Martirosyan found a criterion for the distributivity of the lattice of subvarieties of varieties of alternative and right-alternative algebras.

Recently, Dotsenko and the second author [12] proved that the lattice of subvarieties of a variety  $\mathcal{N}$  of Novikov algebras is distributive if and only if all algebras of the variety  $\mathcal{N}$  satisfy the identities

$$\begin{aligned} \alpha a^2 a + \beta a a^2 &= 0, \\ \gamma((a, a, b) - (b, a, a)) + \delta(a(ab) - ba^2) &= 0 \end{aligned}$$

for some  $((\alpha : \beta), (\gamma : \delta)) \in \mathbb{P}^1 \times \mathbb{P}^1$ .

Let  $\mathfrak{M}$  be a subvariety of the variety of right-commutative algebras defined by the identities

$$a(aa) = 0, \tag{2}$$

$$a[a, b] = 0, \tag{3}$$

$$a(ba) - b(aa) = 0. \tag{4}$$

Let  $\mathfrak{N}$  be a subvariety of the variety of right-commutative algebras defined by the identities (2), and

$$a(ab) - 2a(ba) + b(aa) = 0, \tag{5}$$

$$[a, b]a = 0. \tag{6}$$

In this work, we consider the free algebras  $F_n(\mathfrak{M})$  and  $F_n(\mathfrak{N})$  of rank  $n$  in the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ , respectively. We study the  $S_n$ -module structures of the free algebras  $F_n(\mathfrak{M})$  and  $F_n(\mathfrak{N})$ , and show that the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$  satisfy the distributive conditions. Moreover, we describe all subvarieties of the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ .

In 1950, Malcev [13] and Specht [14] were the first to independently apply the representation theory of the symmetric group to the study of polynomial identity algebras (PI-algebras). In the 1970s, Regev [15] further developed the Malcev–Specht method. In the 1980s, Drensky [16] and Berele [17] began applying the representation theory of the general linear group to the study of PI-algebras.

In 1986, Vladimirova and Drensky [18], using methods from the representation theory of the symmetric and general linear groups, completely described all subvarieties of the variety of associative algebras satisfying an identity of degree three over a field of characteristic zero. In the present paper, we apply the methods developed in [18].

**Proposition III.1.** *As  $S_n$ -modules*

$$P_1(\mathfrak{M}) \cong V(1), \quad P_2(\mathfrak{M}) \cong V(2) \oplus V(1, 1),$$

$$P_3(\mathfrak{M}) \cong V(3) \oplus V(2, 1) \oplus V(1, 1, 1),$$

$$P_n(\mathfrak{M}) \cong V(n) \oplus V(n-1, 1), \quad n \geq 4.$$

The submodules  $V(n)$  and  $V(n-1, 1)$  are generated by the linearizations of  $(\cdots (aa) \cdots)_n a$  and  $(\cdots ([a, b] a) \cdots)_n a$ , respectively.

The submodule  $V(1, 1, 1)$  is generated by  $a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$ .

## II. METHODS

### III. RESULTS AND DISCUSSION

Throughout this paper, the field  $K$  is of zero characteristic. Let  $P_n(\mathfrak{M})$  be a multilinear component of the free algebra  $F_n(\mathfrak{M})$ . Let  $V(\lambda)$  be an irreducible  $S_n$ -module for partition  $\lambda \vdash n$ .

*Proof.* Multiplying (2), (3) and (4) by  $a$  on the left, we obtain

$$a(a(aa)) = 0, \tag{7}$$

$$a(a[a, b]) = 0, \tag{8}$$

$$a(a(ba)) - a(b(aa)) = 0. \tag{9}$$

Multiplying (2) by  $b$  on the left, we get

$$b(a(aa)) = 0. \tag{10}$$

Applying identities (8), (9) and (10) to the expression  $\Delta_{a \rightarrow b} a(a(aa)) = 0$ , we obtain

$$a(b(aa)) = a(a(ba)) = a(a(ab)) = 0. \tag{11}$$

Multiplying identities (3) and (4) by  $b$  on the left, we get

$$b(a(ab)) - b(a(ba)) = 0, \tag{12}$$

$$b(a(ba)) - b(b(aa)) = 0. \tag{13}$$

Interchanging  $a$  and  $b$  in (12) and (13), we get

$$a(b(ba)) - a(b(ab)) = 0, \tag{14}$$

$$a(b(ab)) - a(a(bb)) = 0. \tag{15}$$

Applying identities (12) and (13) to the expression  $\Delta_{a \rightarrow b} b(a(aa)) = 0$ , we obtain

$$b(b(aa)) = 0. \tag{16}$$

Applying (16) to identities (12)-(15), we get

$$a(a(bb)) = a(b(ab)) = b(a(ab)) = a(b(ba)) = b(a(ba)) = 0. \quad (17)$$

Multiplying identities (3) and (4) by  $c$  on the left, we obtain

$$c(a(ab)) - c(a(ba)) = 0, \quad (18)$$

$$c(a(ba)) - c(b(aa)) = 0. \quad (19)$$

It follows from (18) and (19) that

$$c(a(ab)) - c(b(aa)) = 0. \quad (20)$$

After swapping  $b$  and  $c$  in (18)-(20), we get

$$b(a(ac)) - b(a(ca)) = 0, \quad (21)$$

$$b(a(ca)) - b(c(aa)) = 0, \quad (22)$$

$$b(a(ac)) - b(c(aa)) = 0. \quad (23)$$

Applying identities (22) and (23) to the expression  $\Delta_{a \rightarrow c} b(a(aa)) = 0$ , we obtain

$$b(c(aa)) = 0. \quad (24)$$

Applying (24) to identities (18)-(23), we get

$$b(a(ca)) = b(a(ac)) = c(b(aa)) = c(a(ba)) = c(a(ab)) = 0. \quad (25)$$

Applying (25) to the expression  $\Delta_{a \rightarrow c} a(a(ab)) = 0$ , we get

$$a(c(ab)) + a(a(cb)) = 0. \quad (26)$$

Applying (26) to the expression  $\Delta_{b \rightarrow c} a(a(bb)) = 0$ , we get

$$a(c(ab)) - a(a(bc)) = 0. \quad (27)$$

Applying (25) and (26) to the expression  $\Delta_{a \rightarrow c} a[a, b] = 0$ , we get

$$a(c(ba)) + a(a(bc)) = 0. \quad (28)$$

After swapping  $b$  and  $c$  in (26), we obtain,

$$a(b(ac)) + a(a(bc)) = 0. \quad (29)$$

The difference of (28) and (29) is

$$a(c(ba)) - a(b(ac)) = 0. \quad (30)$$

Interchanging  $b$  and  $c$  in (27) and (30), we get

$$a(b(ac)) - a(a(cb)) = 0, \quad (31)$$

$$a(b(ca)) - a(c(ab)) = 0. \quad (32)$$

Applying identities (25), (27) and (30)-(32) to the expression  $\Delta_{a \rightarrow c} (\Delta_{a \rightarrow b} a(a(aa))) = 0$ , we obtain

$$a(a(bc)) + a(c(ab)) = 0. \quad (33)$$

Adding of (27) and (33), we obtain

$$a(c(ab)) = 0. \quad (34)$$

Applying (34) to (27) and (30)-(32), we get

$$a(c(ba)) = a(b(ca)) = a(b(ac)) = a(a(cb)) = a(a(bc)) = 0. \quad (35)$$

Applying (35) to the expression  $\Delta_{b \rightarrow cd} a(ab) - a(ba) = 0$ , we obtain

$$a((cd)a) = 0. \quad (36)$$

From (36) follow

$$a((ba)a) = a((ab)a) = 0. \quad (37)$$

Applying (37) to the expression  $\Delta_{a \rightarrow b} a((aa)a) = 0$ , we obtain

$$b((aa)a) = 0. \quad (38)$$

Applying (36) to the expression  $\Delta_{a \rightarrow b} a((ba)a) = 0$ , we obtain

$$b((ba)a) = 0. \quad (39)$$

Interchanging  $a$  and  $b$  into (39), we get

$$a((ab)b) = 0. \quad (40)$$

Applying (39) to the expression  $\Delta_{a \rightarrow b} b((aa)a) = 0$ , we get

$$b((ab)a) = 0. \quad (41)$$

Applying (36) to the expression  $\Delta_{a \rightarrow c} a((ba)a) = 0$ , we obtain

$$c((ba)a) = 0. \quad (42)$$

Applying (42) to the expression  $\Delta_{a \rightarrow c} b((aa)a) = 0$ , we obtain

$$b((ac)a) = 0. \quad (43)$$

Applying (43) to the expression  $\Delta_{a \rightarrow c} a((ab)a) = 0$ , we get

$$a((ab)c) = 0. \quad (44)$$

Hence,

$$a((bc)d) = 0. \quad (45)$$

Applying (35) and (45) to the expression  $\Delta_{a \rightarrow cd} a(aa) = 0$ , we obtain  $(cd)(aa) = 0$ . Applying the derivation  $\Delta_{a \rightarrow b}$  to the identity  $(cd)(aa) = 0$ , we get

$$(cd)(ab) + (cd)(ba) = 0. \quad (46)$$

Applying (45) and (46) to the expression  $\Delta_{a \rightarrow cd} a(ab) - a(ba) = 0$ , we obtain

$$2(cd)(ab) - a(b(cd)) = 0. \quad (47)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to the identity  $a(a(cd)) = 0$ , we get

$$a(b(cd)) + b(a(cd)) = 0. \quad (48)$$

Applying identities (45) and (48) to the expression  $\Delta_{a \rightarrow cd} a(ba) - b(aa) = 0$ , we get

$$(cd)(ba) + 2a(b(cd)) = 0. \quad (49)$$

Applying (46) to the summing of twice (47) and (49), we obtain

$$(cd)(ab) = 0. \quad (50)$$

Applying (50) to (47), we obtain

$$a(b(cd)) = 0. \quad (51)$$

□

**Proposition III.2.** As  $S_n$ -modules

$$P_1(\mathfrak{N}) \cong V(1), \quad P_2(\mathfrak{N}) \cong V(2) \oplus V(1, 1),$$

$$P_3(\mathfrak{N}) \cong V(3) \oplus V(2, 1) \oplus V(1, 1, 1),$$

$$P_n(\mathfrak{N}) \cong V(0), \quad n \geq 4.$$

The submodules  $V(1)$ ,  $V(2)$  and  $V(1, 1, 1)$  are generated by the linearizations of  $a$ ,  $aa$  and  $ab - ba$ , respectively.

The submodules  $V(3)$ ,  $V(2, 1)$  and  $V(1, 1, 1)$  are generated by the linearizations of  $(aa)a$ ,  $a[a, b]$  and  $a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$ , respectively.

Let  $P_n(\mathfrak{N})$  be a multilinear component of the free algebra  $F_n(\mathfrak{N})$ .

*Proof.* Multiplying (2) by  $a$  on the left, we obtain

$$a(a(aa)) = 0. \quad (52)$$

Multiplying (2) by  $a$  on the right and applying (1), we obtain

$$(aa)(aa) = 0. \quad (53)$$

Applying (52) and (53) to  $\Delta_{a \rightarrow aa}(2)$ , we get

$$a((aa)a) = 0. \quad (54)$$

Applying (1) and (53) to  $\Delta_{b \rightarrow aa}(6)$ , we obtain

$$((aa)a)a = 0. \quad (55)$$

From (6) and (55) follows

$$((ab)c)d = 0. \quad (56)$$

Applying (1) and (56) to  $\Delta_{a \rightarrow aa}(6)$ , we obtain

$$(ab)(aa) - 2(ba)(aa) = 0. \quad (57)$$

Multiplying (2) by  $b$  on the right and applying (1), we get

$$(ab)(aa) = 0. \quad (58)$$

Applying (58) to (57), we obtain

$$(ba)(aa) = 0. \quad (59)$$

Multiplying (5) by  $a$  on the right, and applying (1) and (59), we obtain

$$2(aa)(ba) - (aa)(ab) = 0. \quad (60)$$

Applying (58) and (59) to  $\Delta_{a \rightarrow b}(53)$ , we obtain

$$(aa)(ba) + (aa)(ab) = 0. \quad (61)$$

Summing of (60) and (61), we get

$$(aa)(ba) = 0. \quad (62)$$

Applying (62) to (61), we obtain

$$(aa)(ab) = 0. \quad (63)$$

Applying (58) to  $\Delta_{b \rightarrow ab}(5)$ , we get

$$2a((ab)a) - a(a(ab)) = 0. \quad (64)$$

Applying (58) to  $\Delta_{a \rightarrow ab}(2)$ , we get

$$a((ab)a) + a(a(ab)) = 0. \quad (65)$$

Summing of (64) and (65), we obtain

$$a((ab)a) = 0. \quad (66)$$

Applying (59) to  $\Delta_{b \rightarrow ba}(5)$ , we get

$$2a((ba)a) - a(a(ba)) = 0. \quad (67)$$

Applying (59) to  $\Delta_{b \rightarrow ba}(2)$ , we obtain

$$a((ba)a) + a(a(ba)) = 0. \quad (68)$$

Summing of (67) and (68), we get

$$a((ba)a) = 0. \quad (69)$$

Applying (1), (66) and (69) to  $\Delta_{a \rightarrow b}(54)$ , we obtain

$$b((aa)a) = 0. \quad (70)$$

Applying (66) to (65), we obtain

$$a(a(ab)) = 0. \quad (71)$$

Applying (69) to (68), we obtain

$$a(a(ba)) = 0. \quad (72)$$

Applying (62), (63), (66) and (70), we get

$$2a(b(aa)) - b(a(aa)) = 0. \quad (73)$$

Multiplying (2) by  $a$  on the left, we get

$$b(a(aa)) = 0. \quad (74)$$

Applying (74) to (73), we obtain

$$a(b(aa)) = 0. \quad (75)$$

Multiplying (5) by  $a$  on the right and applying (1), we obtain

$$(ab)(ab) - 2(ab)(ba) + (bb)(aa) = 0. \quad (76)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to (58), we get

$$(ab)(ab) + (ab)(ba) + (bb)(aa) = 0. \quad (77)$$

Subtracting (76) from (77), we get

$$(ab)(ba) = 0. \quad (78)$$

Interchanging  $a$  and  $b$  into (78), we obtain

$$(ba)(ab) = 0. \quad (79)$$

Applying (1), (56) and (79) to the expression  $\Delta_{a \rightarrow ab}(6)$ , we get

$$(ab)(ab) = 0. \quad (80)$$

Interchanging  $a$  and  $b$  into (80), we get

$$(ba)(ba) = 0. \quad (81)$$

Multiplying (6) by  $b$  on the left, we get

$$b((ab)a) - b((ba)a) = 0. \quad (82)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to (70), we get

$$b((ba)a) + 2b((ab)a) = 0. \quad (83)$$

Summing of (82) and (83), we obtain

$$b((ab)a) = 0. \quad (84)$$

Interchanging  $a$  and  $b$  into (84), we get

$$a((ba)b) = 0. \quad (85)$$

Applying (84) to (82), we get

$$b((ba)a) = 0. \quad (86)$$

Interchanging  $a$  and  $b$  into (86), we get

$$a((ab)b) = 0. \quad (87)$$

Multiplying (5) by  $b$  on the left, we get

$$b(a(ab)) - 2b(a(ba)) + b(b(aa)) = 0. \quad (88)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to (74), we obtain

$$b(a(ab)) + b(a(ba)) + b(b(aa)) = 0. \quad (89)$$

Subtracting (88) from (89), we obtain

$$b(a(ba)) = 0. \quad (90)$$

Interchanging  $a$  and  $b$  into (90), we get

$$a(b(ab)) = 0. \quad (91)$$

Applying (78) and (80) to (77), we get

$$(bb)(aa) = 0. \quad (92)$$

Interchanging  $a$  and  $b$  in (92), we get

$$(aa)(bb) = 0. \quad (93)$$

Applying (85) and (92) to  $\Delta_{a \rightarrow bb}a(aa) = 0$ , we obtain

$$a(a(bb)) = 0. \quad (94)$$

Interchanging  $a$  and  $b$  into (94), we get

$$b(b(aa)) = 0. \quad (95)$$

Multiplying (5) by  $c$  on the right and applying (1), we get

$$(ac)(ab) - 2(ac)(ba) + (bc)(aa) = 0. \quad (96)$$

Applying the derivation  $\Delta_{b \rightarrow c}$  to (78), (80) and (92), respectively, we get

$$(ab)(ca) + (ac)(ba) = 0, \quad (97)$$

$$(ab)(ac) + (ac)(ab) = 0, \quad (98)$$

$$(bc)(aa) + (cb)(aa) = 0. \quad (99)$$

Applying (97), (98) and (99) to  $\Delta_{a \rightarrow c}(ab)(aa) = 0$ , we get

$$-(ac)(ab) - (ac)(ba) - (bc)(aa) = 0. \quad (100)$$

Summing of (96) and (100), we get

$$(ac)(ba) = 0. \quad (101)$$

Applying (1), (56) and (101) to  $\Delta_{a \rightarrow ca}(6)$ , we get

$$(ba)(ca) = 0. \quad (102)$$

Applying (1) and (56) to  $\Delta_{a \rightarrow ca}(6)$ , we get

$$(aa)(bc) = 0. \quad (103)$$

Applying (1) and (56) to  $\Delta_{a \rightarrow ac}(6)$ , we get

$$(ab)(ac) - 2(ba)(ac) = 0. \quad (104)$$

Applying (103) to  $\Delta_{a \rightarrow b}(aa)(ac) = 0$ , we obtain

$$(ba)(ac) + (ab)(ac) = 0. \quad (105)$$

Subtracting (104) from (105), we get

$$(ba)(ac) = 0. \quad (106)$$

Applying (106) to (105), we obtain

$$(ab)(ac) = 0. \quad (107)$$

Applying (101) and (107) to (100), we obtain

$$(bc)(aa) = 0. \quad (108)$$

Multiplying (6) by  $c$  on the left, we obtain

$$c((ab)a) - c((ba)a) = 0. \quad (109)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to  $c((aa)a) = 0$ , we get

$$c((ba)a) + 2c((ab)a) = 0. \quad (110)$$

Summing of (109) and (110), we obtain

$$c((ab)a) = 0. \quad (111)$$

Applying (111) to (110), we obtain

$$c((ba)a) = 0. \quad (112)$$

Applying (1) to  $\Delta_{b \rightarrow c}(87)$ , we get

$$a((ab)c) = 0. \quad (113)$$

Applying (108) to  $\Delta_{a \rightarrow bc}(2)$ , we obtain

$$a((bc)a) + a(a(bc)) = 0. \quad (114)$$

Applying (108) to  $\Delta_{a \rightarrow bc}(5)$ , we obtain

$$2a((bc)a) - a(a(bc)) = 0. \quad (115)$$

Summing of (114) and (115), we get

$$a((bc)a) = 0. \quad (116)$$

Applying (116) to (114), we obtain

$$a(a(bc)) = 0. \quad (117)$$

Applying (101), (103), (107) and (111) to  $\Delta_{a \rightarrow ac}(5)$ , we obtain

$$2a(b(ac)) - b(a(ac)) = 0. \quad (118)$$

Applying (117) to  $\Delta_{a \rightarrow b}a(a(ac)) = 0$ , we get

$$b(a(ac)) + a(b(ac)) = 0. \quad (119)$$

Summing of (118) and (119), we obtain

$$a(b(ac)) = 0. \quad (120)$$

Applying (120) to (119), we obtain

$$b(a(ac)) = 0. \quad (121)$$

Applying (102), (106), (112) and (116) to  $\Delta_{a \rightarrow ca}(5)$ , we get

$$2a(b(ca)) - b(a(ca)) = 0. \quad (122)$$

Applying (117) to  $\Delta_{a \rightarrow ba}(a(ca)) = 0$ , we obtain

$$a(b(ca)) + b(a(ca)) = 0. \quad (123)$$

Summing of (122) and (123), we obtain

$$a(b(ca)) = 0. \quad (124)$$

Applying (124) to (123), we obtain

$$b(a(ca)) = 0. \quad (125)$$

Applying (121) and (125) to  $\Delta_{a \rightarrow bc}(a(aa)) = 0$ , we obtain

$$c(b(aa)) = 0. \quad (126)$$

Applying (1) and (56) to  $\Delta_{a \rightarrow cd}(6)$ , we obtain

$$2(ba)(cd) - (ab)(cd) = 0. \quad (127)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to  $(aa)(cd) = 0$ , we get

$$(ba)(cd) + (ab)(cd) = 0. \quad (128)$$

Summing of (127) and (128), we obtain

$$(ba)(cd) = 0. \quad (129)$$

From (54), (66), (69), (70), (84)-(87), (111)-(113), (116) follows

$$a((bc)d) = 0. \quad (130)$$

Applying (129) and (130) to  $\Delta_{a \rightarrow cd}(5)$ , we get

$$2a(b(cd)) - b(a(cd)) = 0. \quad (131)$$

Applying the derivation  $\Delta_{a \rightarrow b}$  to  $a(a(cd)) = 0$ , we get

$$a(b(cd)) + b(a(cd)) = 0. \quad (132)$$

Summing of (131) and (132), we obtain

$$a(b(cd)) = 0. \quad (133)$$

□

**Theorem III.1.** Let  $f \in F_n(\mathfrak{M})$ . Then the consequences of higher degree of  $f$  are equivalent to the following identities

$$\begin{aligned} & a)a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba), \text{ if } f = aa; \\ & b)a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba), \text{ if } f = ab - ba; \\ & c)\underbrace{\dots((aa)a \dots)}_{n+1}a \text{ and } \underbrace{\dots([a, b]a \dots)}_{n-1}a, \text{ if } f = \underbrace{\dots((aa)a \dots)}_n a, n \geq 1; \\ & d)\underbrace{\dots((aa)a \dots)}_{n+1}a \text{ and } \underbrace{\dots([a, b]a \dots)}_{n-1}a, \text{ if } f = \underbrace{\dots([a, b]a \dots)}_{n-2} a, n \geq 2. \end{aligned}$$

*Proof.* a) Applying  $aa = 0$  to (4), we get

$$(ba)a = 0.$$

Applying the derivation  $\Delta_{a \rightarrow c}$  to the identity  $(ba)a = 0$ , we obtain

$$(bc)a + (ba)c = 0.$$

Applying (1) to the identity  $(bc)a + (ba)c = 0$ , we get

$$(bc)a = 0.$$

Since  $a(bc) = -(bc)a$ . Hence,

$$a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba) = 0.$$

b) This follows from identity  $ab - ba = 0$ .

c)  $(\dots \underbrace{(aa)a}_{n+1} \dots)a = ((\dots \underbrace{(aa)a}_{n} \dots)a)a = 0$ .

Let  $n = 1$ . It is clear.

Let  $n > 1$ . Then

$$((\dots \underbrace{(aa)a}_{n} \dots)a)b = 0.$$

Applying the last identity and (1) to the expression  $\Delta_{a \rightarrow b}(\dots \underbrace{(aa)a}_{n+1} \dots)a = 0$ , we obtain

$$(\dots \underbrace{(ba)a}_{n+1} \dots)a = 0.$$

Hence,  $(\dots \underbrace{[a, b]a}_{n-1} \dots)a = 0$ .

d)  $(\dots \underbrace{[a, b]a}_{n-1} \dots)a = ((\dots \underbrace{[a, b]a}_{n-2} \dots)a)a = 0$ .

Applying (2) to the expression  $\Delta_{b \rightarrow aa}(\dots \underbrace{[a, b]a}_{n-2} \dots)a = 0$ , we obtain

$$(\dots \underbrace{(aa)a}_{n+1} \dots)a = 0.$$

□

**Theorem III.2.** Let  $f \in F_n(\mathfrak{N})$ . Then the consequences of higher degree of  $f$  are equivalent to the following identities

a)  $aa, ab - ba$ , if  $f = a$ ;

b)  $(aa)a, a[a, b], a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$ , if  $f = aa$ ;

c)  $a[a, b], a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$ , if  $f = ab - ba$ .

*Proof.* a) It is clear.

b) From  $(aa)a = 0$  and (6) follows  $(ab)c = 0$ . Applying the last identity to the expression  $\Delta_{a \rightarrow bc}aa = 0$ , we obtain  $a(bc) = 0$ .

c) This follows from identity  $ab - ba = 0$ . □

#### IV. CONCLUSION AND FUTURE WORK

In this paper, we studied the free algebras in the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ . We investigated the  $S_n$ -module structures of the free algebras  $F_n(\mathfrak{M})$  and  $F_n(\mathfrak{N})$ . As a consequence, we proved that the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$  satisfy the distributivity conditions. Moreover, we described all subvarieties of the varieties  $\mathfrak{M}$  and  $\mathfrak{N}$ .

It would be of interest to investigate whether there exist further subvarieties of the variety of right-commutative algebras satisfying the distributivity conditions.

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