

Article

Varieties of right-commutative algebras defined by identities $(aa)a = [a, b]a = 0$

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Abstract

An algebra R is called a *right-commutative* algebra if it satisfies the right-commutative identity

$$(x_1x_2)x_3 - (x_1x_3)x_2 = 0,$$

for all elements $x_1, x_2, x_3 \in R$.

This algebra is well known by other name, such as *NAP* algebra or a *non-associative permutative* algebra. Varieties of right-commutative algebras contain important subvarieties such as Novikov and bicommutative algebras.

In this work, we consider two subvarieties $\mathfrak{M}$ and $\mathfrak{N}$ of the variety of right-commutative algebras. The subvariety $\mathfrak{M}$ is defined by the identities $(x_1x_1)x_1 = 0$, $[x_1, x_2]x_1 = 0$ and $x_1(x_2x_1) - x_2(x_1x_1) = 0$. The subvariety $\mathfrak{N}$ is defined by the identities $(x_1x_1)x_1 = 0$, $[x_1, x_2]x_1 = 0$ and $x_1(x_1x_2) - 2x_1(x_2x_1) + x_2(x_1x_1) = 0$.

We study the free algebras in the varieties $\mathfrak{M}$ and $\mathfrak{N}$ of right-commutative algebras. We investigate the S_n -module structures of the free algebras $F_n(\mathfrak{M})$ and $F_n(\mathfrak{N})$ in the varieties $\mathfrak{M}$ and $\mathfrak{N}$, respectively. As a consequence, we prove that the subvarieties $\mathfrak{M}$ and $\mathfrak{N}$ of the variety of right-commutative algebras satisfy the distributivity conditions. Moreover, we describe all subvarieties of the varieties $\mathfrak{M}$ and $\mathfrak{N}$.

Keywords: variety of algebras, lattice of subvarieties, free algebra, right-commutative algebra, S_n -module structure

I. INTRODUCTION

An algebra R is called a *right-commutative* algebra if for all elements $x_1, x_2, x_3 \in R$

$$(x_1x_2)x_3 - (x_1x_3)x_2 = 0. \quad (1)$$

This algebra is well known by other name, such as *NAP* algebra or *non-associative permutative* algebra, see [1], [2]. If we consider the variety of NAP algebras, the varieties of left-Novikov algebras and bicommutative algebras (or LR-algebras) are subclasses of

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the class of NAP algebras. These classes of algebras have been widely studied. For example, in [3], a free basis was constructed and the codimensions of Novikov algebras were found. Moreover, it was shown that the Novikov operad is not Koszul. In [4], the S_n -module structures of free Novikov algebras were investigated.

LR-algebras first appeared in the work of Dzhumadil'daev and Tulenbaev [5]. In [6], a free basis was constructed, the codimensions were calculated, and the S_n -module structures of free LR-algebras were fully studied. In 2018, Drensky and Zhakhayev [7] gave an affirmative answer to the Specht problem for bicommutative algebras.

In 1976, Bokut [8] formulated the following question: to describe, in terms of identities, the varieties of algebras whose lattice of subvarieties is distributive. In 1976, Ananin and Kemer [2] proved that the lattice of subvarieties of the variety $\mathcal{A}$ of associative algebras over a field K of characteristic zero is distributive if and only if the T -ideal of identities of $\mathcal{A}$ contains an identity of the form

$$\alpha[x, y]y + \beta y[x, y] = 0,$$

where $\alpha, \beta \in K$ and $(\alpha, \beta) \neq (0, 0)$.

In [10] and [11], Martirosyan found a criterion for the distributivity of the lattice of subvarieties of varieties of alternative and right-alternative algebras.

Recently, Dotsenko and the second author [12] proved that the lattice of subvarieties of a variety $\mathcal{N}$ of Novikov algebras is distributive if and only if all algebras of the variety $\mathcal{N}$ satisfy the identities

$$\begin{aligned} \alpha a^2 a + \beta a a^2 &= 0, \\ \gamma((a, a, b) - (b, a, a)) + \delta(a(ab) - ba^2) &= 0 \end{aligned}$$

for some $((\alpha : \beta), (\gamma : \delta)) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathfrak{M}$ be a subvariety of the variety of right-commutative algebras defined by the identities

$$(aa)a = 0, \tag{2}$$

$$[a, b]a = 0, \tag{3}$$

$$a(ba) - b(aa) = 0. \tag{4}$$

Let $\mathfrak{N}$ be a subvariety of the variety of right-commutative algebras defined by the identities (2), (3) and

$$a(ab) - 2a(ba) + b(aa) = 0. \tag{5}$$

In this work, we consider the free algebras $F_n(\mathfrak{M})$ and $F_n(\mathfrak{N})$ of rank n in the variety $\mathfrak{M}$ and $\mathfrak{N}$, respectively. We study the S_n -module structures of the free algebras $F_n(\mathfrak{M})$ and $F_n(\mathfrak{N})$, and show that the varieties $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the distributive conditions. Moreover, we describe all subvarieties of the varieties $\mathfrak{M}$ and $\mathfrak{N}$.

II. METHODS

In 1950, Malcev [13] and Specht [14] were the first to independently apply the representation theory of the symmetric group to the study of polynomial identity algebras (PI-algebras). In the 1970s, Regev [15] further developed the Malcev–Specht method. In the 1980s, Drensky [16] and Berele [17] began applying the representation theory of the general linear group to the study of PI-algebras.

In 1986, Vladimirova and Drensky [18], using methods from the representation theory of the symmetric and general linear groups, completely described all subvarieties of the variety of associative algebras satisfying an identity of degree three over a field of characteristic zero. In the present paper, we apply the methods developed in [18].

Let $X_n = \{x_1, \dots, x_n\}$ be a set of generators, and let $K\{X_n\}$ be the absolutely free algebra on X_n . Let $\Delta_{x_i \mapsto x_j}$ be the derivation of $K\{X_n\}$ defined by

$$\begin{aligned}\Delta_{x_i \mapsto x_j}(x_i) &= x_j, \\ \Delta_{x_i \mapsto x_j}(x_k) &= 0, \quad k \neq i.\end{aligned}$$

III. RESULTS AND DISCUSSION

Throughout this paper, the field K is of characteristic zero. Let $P_n(\mathfrak{M})$ be a multilinear part of the free algebra $F_n(\mathfrak{M})$. Let $V(\lambda)$ be an irreducible S_n -module for partition $\lambda \vdash n$.

Proposition III.1. *As S_n -modules*

$$\begin{aligned}P_1(\mathfrak{M}) &\cong V(1), \quad P_2(\mathfrak{M}) \cong V(2) \oplus V(1, 1), \\ P_3(\mathfrak{M}) &\cong V(3) \oplus V(2, 1) \oplus V(1, 1, 1), \\ P_n(\mathfrak{M}) &\cong V(n) \oplus V(n-1, 1), \quad n \geq 4.\end{aligned}$$

The submodules $V(n)$ and $V(n-1, 1)$ are generated by the linearizations of $\underbrace{a(\dots(aa)\dots)}_n$ and $\underbrace{a(\dots(a[a, b])\dots)}_{n-2}$, respectively.

The submodule $V(1, 1, 1)$ is generated by $a(bc) - a(cb) + c(ab) - b(ac) + \underbrace{b(ca)}_n - \underbrace{c(ba)}_{n-2}$.

Proof. From (2) and (3) follows

$$(ab)c = 0.$$

From the last identity directly follow

$$((ab)c)d = 0, \tag{6}$$

$$(ab)(cd) = 0, \tag{7}$$

$$d((ab)c) = 0. \tag{8}$$

Applying (7) and (8) to the expression

$$\Delta_{a \mapsto aa} a(ba) - b(aa) = 0$$

we get

$$a(b(aa)) - b(a(aa)) = 0.$$

Multiplying (4) by a on the left, we get

$$a(a(ba)) - a(b(aa)) = 0.$$

Multiplying (4) by b on the left, we obtain

$$b(a(ba)) - b(b(aa)) = 0. \tag{9}$$

After swapping a and b in (9), we get

$$a(b(ab)) - a(a(bb)) = 0. \tag{10}$$

Applying (7) and (8) to the expression

$$\Delta_{a \mapsto ab} a(ba) - b(aa) = 0$$

we get

$$a(b(ab)) - b(a(ab)) = 0. \tag{11}$$

After swapping a and b in (11), we get

$$b(a(ba)) - a(b(ba)) = 0. \tag{12}$$

Multiplying (4) by c on the left, we obtain

$$c(a(ba)) - c(b(aa)) = 0. \quad (13)$$

Interchanging elements b and c in (13), we get

$$b(a(ca)) - b(c(aa)) = 0. \quad (14)$$

Using (7) and (8) in the expression

$$\Delta_{a \mapsto cd} a(ba) - b(aa) = 0$$

we get

$$a(b(cd)) - b(a(cd)) = 0. \quad (15)$$

From (13), (14) and (15) follow

$$a(b(ca)) = a(c(ba)) = b(a(ca)) = b(c(aa)) = c(a(ba)) = c(b(aa)). \quad (16)$$

Applying (16) to the expression

$$\Delta_{a \mapsto c} a(a(ba)) - a(b(aa)) = 0$$

we obtain

$$a(a(bc)) - a(b(ac)) = 0. \quad (17)$$

Using (15) in the expression

$$\Delta_{a \mapsto d} a(a(bc)) - a(b(ac)) = 0$$

we get

$$2a(d(bc)) - d(b(ac)) - a(b(dc)) = 0. \quad (18)$$

Interchanging elements a and b in (18), we get

$$2b(d(ac)) - d(a(bc)) - b(a(dc)) = 0. \quad (19)$$

Using (15) in the difference of (18) and (19), we obtain

$$a(d(bc)) - b(d(ac)) = 0. \quad (20)$$

As a result, from (15) and (20) follow

$$c(b(ad)) = c(a(bd)) = b(c(ad)) = b(a(cd)) = a(c(bd)) = a(b(cd)). \quad (21)$$

□

Let $P_n(\mathfrak{N})$ be a multilinear component of the free algebra $F_n(\mathfrak{N})$.

Proposition III.2. *As S_n -modules*

$$P_1(\mathfrak{N}) \cong V(1), \quad P_2(\mathfrak{N}) \cong V(2) \oplus V(1, 1),$$

$$P_3(\mathfrak{N}) \cong V(3) \oplus V(2, 1) \oplus V(1, 1, 1),$$

$$P_n(\mathfrak{N}) \cong V(0), \quad n \geq 4.$$

The submodules $V(1)$, $V(2)$ and $V(1, 1)$ are generated by the linearizations of a , aa and $ab - ba$, respectively.

The submodules $V(3)$, $V(2, 1)$ and $V(1, 1, 1)$ are generated by the linearizations of $a(aa)$, $a(ab) - a(ba)$ and $a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$, respectively. .

Proof. From (2) and (3) follows

$$(ab)c = 0.$$

From the last identity directly follow

$$((ab)c)d = a((bc)d) = (ab)(cd) = 0. \quad (22)$$

Applying (22) to the expression $\Delta_{b \mapsto aa}(5)$, we get

$$a(a(aa)) = 0. \quad (23)$$

Applying (22) to the expression $\Delta_{b \mapsto ab}(5)$, we obtain

$$a(a(ab)) = 0. \quad (24)$$

Applying (22) to the expression $\Delta_{b \mapsto ba}(5)$, we obtain

$$a(a(ba)) = 0. \quad (25)$$

Applying (22) to the expression $\Delta_{a \mapsto aa}(5)$, we obtain

$$2a(b(aa)) - b(a(aa)) = 0. \quad (26)$$

Applying (24) and (25) to the expression $\Delta_{a \mapsto b}(23)$, we obtain

$$a(b(aa)) + b(a(aa)) = 0. \quad (27)$$

Summing of (26) and (27), we get

$$a(b(aa)) = 0. \quad (28)$$

Applying (28) to (27), we obtain

$$b(a(aa)) = 0. \quad (29)$$

Applying (22) to the expression $\Delta_{b \mapsto bb}(5)$, we get

$$a(a(bb)) = 0. \quad (30)$$

Applying (22) to the expression $\Delta_{a \mapsto ab}(5)$, we get

$$2a(b(ab)) - b(a(ab)) = 0. \quad (31)$$

Using (30) in the expression $\Delta_{a \mapsto b}(24)$, we get

$$a(b(ab)) + b(a(ab)) = 0. \quad (32)$$

Summing of (31) and (32), we obtain

$$a(b(ab)) = 0. \quad (33)$$

Applying (33) to (32), we get

$$b(a(ab)) = 0. \quad (34)$$

After swapping a and b in (30), (33) and (34), we get, respectively

$$b(b(aa)) = 0, \quad (35)$$

$$b(a(ba)) = 0, \quad (36)$$

$$a(b(ba)) = 0. \quad (37)$$

Applying (22) to the expression $\Delta_{b \mapsto bc}(5)$, we get

$$a(a(bc)) = 0. \quad (38)$$

Applying (22) to the expression $\Delta_{a \mapsto ac}(5)$, we get

$$2a(b(ac)) - b(a(ac)) = 0. \quad (39)$$

Applying (38) to the expression $\Delta_{a \mapsto b} a(ac) = 0$, we obtain

$$b(a(ac)) + a(b(ac)) = 0. \quad (40)$$

Summing of (39) and (40), we get

$$a(b(ac)) = 0. \quad (41)$$

Applying (41) to (40), we get

$$b(a(ac)) = 0. \quad (42)$$

Using (22) in the expression $\Delta_{a \mapsto ca}(5)$, we get

$$2a(b(ca)) - b(a(ca)) = 0. \quad (43)$$

Applying (38) to the expression $\Delta_{a \mapsto b} a(ca) = 0$, we obtain

$$b(a(ca)) + a(b(ca)) = 0. \quad (44)$$

Summing of (43) and (44), we get

$$a(b(ca)) = 0. \quad (45)$$

Applying (45) to (44), we obtain

$$b(a(ca)) = 0. \quad (46)$$

Applying (46) and (42) to the expression $\Delta_{a \mapsto c} b(aa) = 0$, we get

$$b(c(aa)) = 0. \quad (47)$$

Applying (22) to $\Delta_{a \mapsto cd}(5)$, we get

$$2a(b(cd)) - b(a(cd)) = 0. \quad (48)$$

Using the derivation $\Delta_{a \mapsto b}$ in the expression $a(cd) = 0$, we obtain

$$a(b(cd)) + b(a(cd)) = 0. \quad (49)$$

Summing of (48) and (49), we obtain

$$a(b(cd)) = 0.$$

□

Theorem III.1. Let $f \in F_n(\mathfrak{M})$. Then the consequences of higher degree of f are equivalent to the following identities

- a) $a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$, if $f = aa$;
- b) $a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba)$, if $f = ab - ba$;
- c) $\underbrace{a(\cdots(a(aa))\cdots)}_{n+1}$ and $\underbrace{a(\cdots(a[a,b])\cdots)}_{n-1}$, if $f = \underbrace{a(\cdots(a(aa))\cdots)}_n$, $n \geq 1$;
- d) $\underbrace{a(\cdots(a(aa))\cdots)}_{n+1}$ and $\underbrace{a(\cdots(a[a,b])\cdots)}_{n-1}$, if $f = \underbrace{a(\cdots(a[a,b])\cdots)}_{n-2}$, $n \geq 2$.

Proof. a) Applying $aa = 0$ to (4), we get

$$(ba)a = 0.$$

Using the derivation $\Delta_{a \mapsto c}$ in the identity $(ba)a = 0$, we obtain

$$(bc)a + (ba)c = 0.$$

Applying (1) to the identity $(bc)a + (ba)c = 0$, we get

$$(bc)a = 0.$$

Since $a(bc) = -(bc)a$. Hence,

$$a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba) = 0.$$

b) It is clear.

$$c) \underbrace{a(\cdots(aa)\cdots)}_{n+1} = a(\underbrace{a(\cdots(aa)\cdots)}_n) = 0.$$

Let $n = 1$. It is clear.

Let $n = 2$. Then from $aa = 0$ follows $b(aa) = 0$. From identities $b(aa) = 0$ and (4) follows $a(ba) = 0$. Applying identities $b(aa) = 0$ and $a(ba) = 0$ to the expression

$$\Delta_{a \mapsto b} a(aa) = 0$$

we obtain

$$a(ab) = 0.$$

Hence, $a[a, b] = a(ab) - a(ba) = 0$.

Let $n > 2$. Applying identities (21) and $b(\underbrace{a(\cdots(aa)\cdots)}_n) = 0$ to the expression

$$\Delta_{a \mapsto b} \underbrace{a(\cdots(aa)\cdots)}_{n+1} = 0$$

we obtain

$$a(\cdots(ab)\cdots) = 0$$

Hence, $\underbrace{a(\cdots(a[a, b])\cdots)}_{n-1} = 0$.

$$d) \underbrace{a(\cdots(a[a, b])\cdots)}_{n-1} = a(\underbrace{a(\cdots(a[a, b])\cdots)}_{n-2}) = 0.$$

Applying identities (2) and (21) to the expression

$$\Delta_{b \mapsto aa} \underbrace{a(\cdots(a[a, b])\cdots)}_{n-2} = 0$$

we get

$$\underbrace{a(\cdots(aa)\cdots)}_{n+1} = 0.$$

□

Theorem III.2. Let $f \in F_n(\mathfrak{N})$. Then the consequences of higher degree of f are equivalent to the following identities

$$a) aa, ab - ba, \text{ if } f = a;$$

$$b) a(aa), a[a, b], a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba), \text{ if } f = aa;$$

$$c) a[a, b], a(bc) - a(cb) + c(ab) - b(ac) + b(ca) - c(ba), \text{ if } f = ab - ba.$$

Proof. a) It is clear.

b) From (2) and (3) follows $(ab)c = 0$. Applying the last identity to the expression $\Delta_{a \mapsto bc} aa = 0$, we obtain $a(bc) = 0$.

c) It is clear.

□

IV. CONCLUSION AND FUTURE WORK

In this work, we studied the free algebras in the varieties $\mathfrak{M}$ and $\mathfrak{N}$. We investigated the S_n -module structures of the free algebras $F_n(\mathfrak{M})$ and $F_n(\mathfrak{N})$. As a consequence, we proved that the varieties $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the distributivity conditions. Moreover, we described all subvarieties of the varieties $\mathfrak{M}$ and $\mathfrak{N}$.

It would be of interest to investigate whether there exist further subclasses of the class of right-commutative algebras satisfying the distributivity conditions. Moreover, it would be interesting to study operads encoding the above mentioned subvarieties of the variety of right-commutative algebras.

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