

Article

Space of Tortken elements in free Novikov algebra

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Abstract

In this paper, we study the one-generated case of the space of Tortken elements in a free Novikov algebra. Using the differential-monomial realization, we prove that the dimensions of the homogeneous components are given by $\dim T_1 = \dim T_2 = 1$ and $\dim T_n = p(n-2)$ for all $n \geq 3$, where $p(n-2)$ denotes the partition number. We also describe the first homogeneous components in the circle language and analyze the relations among their elements in low degrees. In particular, we show that the elements are linearly independent up to degree 5, while the first nontrivial relation appears in degree 6. We derive explicit identities in degrees 6 and 7 and prove a recursive inclusion for homogeneous components in the one-generated case for $n \leq 7$. These results give a more precise description of the low-degree structure of the space of Tortken elements and suggest a regular pattern that may persist in higher degrees. We also formulate a conjecture extending the recursive inclusion to all degrees.

Keywords: Novikov algebra, Tortken algebra, symmetric elements, free algebra

I. INTRODUCTION

An algebra $(A, \cdot)$ is called a (right) Novikov algebra if, for all $a, b, c \in A$, the following identities hold:

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad (1)$$

$$(a \cdot b) \cdot c - a \cdot (b \cdot c) = (a \cdot c) \cdot b - a \cdot (c \cdot b). \quad (2)$$

The first identity (1) is called the **left-commutativity** identity, and the second identity (2) is called the **right-symmetric** identity, and any algebra satisfying this identity is called the right-symmetric algebra. Thus, throughout the paper we use the convention that a (right) Novikov algebra is an algebra satisfying identities (1) and (2).

Example 1.1. Let us consider the polynomial algebra $A = K[x]$ with multiplication

$$a \cdot b = D(a)b$$

A direct verification shows that $(A, \cdot)$ is a Novikov algebra.

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Now, let us define a new product on the Novikov algebra, which is called a symmetric product (anti-commutator). Let A be a (right) Novikov algebra with multiplication $a \cdot b$. Then the anti-commutator is defined by:

$$a \circ b := a \cdot b + b \cdot a$$

Notice that $a \circ b = b \circ a$.

Dzhumadil'daev [1], [3] proved that the symmetrized product satisfies the following identities:

$$(a \circ b) \circ (c \circ d) - (a \circ d) \circ (c \circ b) = (a, b, c)^+ \circ d - (a, d, c)^+ \circ b, \quad (3)$$

$$\begin{aligned} & (((a \circ a) \circ a) \circ b) \circ b + (((a \circ b) \circ b) \circ a) \circ a \\ & + 2(((a \circ a) \circ b) \circ b) \circ a + 2(((a \circ b) \circ a) \circ a) \circ b \\ & - 3(((a \circ a) \circ b) \circ a) \circ b - 3(((a \circ b) \circ a) \circ b) \circ a = 0 \end{aligned} \quad (4)$$

where

$$(a, b, c)^+ = a \circ (b \circ c) - (a \circ b) \circ c.$$

The first identity is called the Tortken identity, and the second identity is the Besken identity. Algebras that satisfy (3) are called Tortken algebras.

Let us define the free Novikov algebra generated by a set X and call it $Nov\langle X \rangle$. Then the space of symmetric elements $S\langle X \rangle$ is the smallest subspace of $Nov\langle X \rangle$ that contains X and is closed under the $\circ$ -product. Elements of $S\langle X \rangle$ are called symmetric elements or Tortken elements, otherwise.

Now, let us define $\mathcal{M}$ as a set of all differential monomials with one derivation. A basis of the free Novikov algebras was constructed by Dzhumadil'daev and Löfwall [2]:

Theorem I.2. *The set*

$$\mathcal{N} = \{ u \in \mathcal{M} \mid \deg(u) - d(u) = 1 \}$$

is a basis of the free Novikov algebra $Nov\langle X \rangle$.

And recently, Dzhumadil'daev and Ismailov [4] constructed a basis of the space of Tortken elements in the free Novikov algebra in terms of differential polynomials:

Theorem I.3. *The set*

$$\overline{\mathcal{M}} := X \cup \{ u' \mid u \in \mathcal{M}, \deg(u) - d(u) = 2 \}$$

is a basis of the space of Tortken elements $S\langle X \rangle$.

In this paper, we investigate the space of Tortken elements in a free Novikov algebra in the one-generated case. While a basis in terms of differential monomials is already known [4], the corresponding description in terms of $\circ$ -product remains unclear. Our aim is to describe this basis in low degrees and to compare it with the known differential-monomial model. We show that the dimensions of the homogeneous components are governed by the partition numbers $p(n-2)$, and we construct explicit basis elements in the $\circ$ -language for small degrees. The necessary identities are verified up to degree seven by means of computations carried out in *Wolfram Mathematica*. This work may be regarded as a step toward the study of free special Tortken algebras in the $\circ$ -product language.

II. METHODS

Proposition II.1. *Let T_n be the homogeneous component of degree n of the space of Tortken elements in the free Novikov algebra generated by one element x . Then*

$$\dim T_1 = \dim T_2 = 1, \quad \dim T_n = p(n-2) \quad \text{for all } n \geq 3,$$

where $p(m)$ denotes the number of partitions of m .

Proof. By Theorem I.3, a basis of $S\langle x \rangle$ is given by

$$\overline{\mathcal{M}} := X \cup \{ u' \mid u \in \mathcal{M}, \deg(u) - d(u) = 2 \},$$

where $\mathcal{M}$ is the set of differential monomials. In the one-generated case, every differential monomial can be written uniquely in the form

$$u = x^{(i_1)} x^{(i_2)} \dots x^{(i_n)}$$

Moreover,

$$\deg(u) = n, \quad d(u) = i_1 + \dots + i_n.$$

Hence the condition $\deg(u) - d(u) = 2$ becomes

$$i_1 + \dots + i_n = n - 2.$$

Since all factors involve the same generator x and the product is commutative, we may assume without loss of generality that the derivation orders are arranged in nonincreasing order:

$$i_1 \geq i_2 \geq \dots \geq i_n \geq 0.$$

Then the number such nonincreasing sequences is equal to the number of partitions of $n - 2$. Therefore,

$$\dim T_n = p(n - 2), \quad n \geq 2.$$

Finally, for $n = 1$, the homogeneous component T_1 is spanned by x . For $n = 2$, the component T_2 is spanned by $x \circ x$. Hence,

$$\dim T_1 = \dim T_2 = 1.$$

This proves the claim. □

Proposition II.2. *Let T_n be the homogeneous component of degree n in the one-generated case. Then,*

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \quad n \leq 7.$$

Proof. In the one-generator case, bases in the $\circ$ -language for the first degrees may be chosen as follows:

$$T_1 = \langle x \rangle, \quad T_2 = \langle x \circ x \rangle, \quad T_3 = \langle (x \circ x) \circ x \rangle,$$

The first three degrees are immediate, since each homogeneous component is one-dimensional.

A. Degree $n = 4$

Degree 4 is the first case in which the homogeneous component has more than one basis element, so we write it out in detail. Let

$$w_1 = ((x \circ x) \circ x) \circ x, \quad w_2 = (x \circ x) \circ (x \circ x).$$

The differential basis of T_4 is given by

$$b_1 = (x''xxx)' = x'''xxx + 3x''x'xx, \quad b_2 = (x'x'xx)' = 2x''x'xx + 2x'x'x'x.$$

Using the identity $a \circ b = (ab)'$, we obtain

$$w_1 = ((x \circ x) \circ x) \circ x = 2x'''xxx + 14x''x'xx + 8x'x'x'x$$

$$w_2 = (x \circ x) \circ (x \circ x) = 8x''x'xx + 8x'x'x'x$$

$$2x'''xxx + 14x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \Rightarrow \alpha_1 = 2, \quad \alpha_2 = 4$$

$$8x''x'xx + 8x'x'x'x = \alpha_1 b_1 + \alpha_2 b_2 \Rightarrow \alpha_1 = 0, \quad \alpha_2 = 4$$

That is,

$$w_1 = 2b_1 + 4b_2, \quad w_2 = 4b_2.$$

w_1 and w_2 are linearly independent.

B. Degree $n = 5$

Since

$$\dim T_5 = p(3) = 3$$

in degree $n = 5$, we have three basis elements which are

$$w_1 = (((x \circ x) \circ x) \circ x) \circ x, \quad w_2 = ((x \circ x) \circ (x \circ x)) \circ x, \quad w_3 = ((x \circ x) \circ x) \circ (x \circ x).$$

The differential basis of T_5 is given by

$$b_1 = (x'''xxx)', \quad b_2 = (x''x'xxx)', \quad b_3 = (x'x'x'xx)'$$

Therefore, we compute

$$w_1 = 2b_1 + 14b_2 + 8b_3, \quad w_2 = 8b_2 + 8b_3, \quad w_3 = 4b_2 + 8b_3$$

Here, w_1, w_2 and w_3 are linearly independent.

C. Degree $n = 6$

The first degree where linear dependence among natural $\circ$ -words appears is degree 6. Consider the six natural commutative nonassociative words

$$\begin{aligned} w_1 &= (((x \circ x) \circ x) \circ x) \circ x, \\ w_2 &= (((x \circ x) \circ (x \circ x)) \circ x) \circ x, \\ w_3 &= (((x \circ x) \circ x) \circ (x \circ x)) \circ x, \\ w_4 &= (((x \circ x) \circ x) \circ x) \circ (x \circ x), \\ w_5 &= ((x \circ x) \circ (x \circ x)) \circ (x \circ x), \\ w_6 &= (x \circ (x \circ x)) \circ (x \circ (x \circ x)). \end{aligned}$$

Since

$$\dim T_6 = p(4) = 5,$$

these six elements cannot be linearly independent.

To construct the identity in degree 6, we use *Wolfram Mathematica*. As a first step, we implement the derivation operator and the symmetrized product in the commutative differential algebra by encoding their basic properties; see Appendix A.

Then we enumerate all six elements of degree 6 in the $\circ$ -language and list all seven basis elements of the homogeneous component of degree 6 of the free Novikov algebra on one generator and assign to each element w_i a scalar coefficient λ_i and consider the linear combination

$$\sum_{i=1}^6 \lambda_i w_i = 0.$$

The program then determines the relations among the coefficients λ_i and computes their admissible values; see Appendix B.

The resulting relations among the coefficients in the case $n = 6$ are

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_3 &= -\frac{\lambda_2}{2}, \\ \lambda_4 &= -\frac{\lambda_2}{2}, \\ \lambda_5 &= \frac{\lambda_2}{2}, \\ \lambda_6 &= -\frac{\lambda_2}{2}. \end{aligned}$$

Taking $\lambda_2 = 2$, we obtain the following identity:

$$\begin{aligned} & 2((x \circ x) \circ (x \circ x)) \circ (x \circ x) - ((x \circ x) \circ x) \circ ((x \circ x) \circ x) \\ & - (((x \circ x) \circ x) \circ x) \circ (x \circ x) + (((x \circ x) \circ (x \circ x)) \circ x) \circ x \\ & - (((x \circ x) \circ x) \circ (x \circ x)) \circ x = 0 \end{aligned} \quad (5)$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

This identity is a consequence of the Tortken identity with

$$a = d = x \circ x, \quad b = c = x.$$

Substituting these values into the Tortken identity gives

$$((x \circ x) \circ x) \circ (x \circ (x \circ x)) - ((x \circ x) \circ (x \circ x)) \circ (x \circ x) = (x \circ x, x, x)^+ \circ (x \circ x) - (x \circ x, x \circ x, x)^+ \circ x.$$

Indeed, this substitution gives

$$-2w_5 + w_6 + w_4 - w_2 + w_3 = 0,$$

or equivalently,

$$2w_5 - w_6 - w_4 + w_2 - w_3 = 0.$$

Therefore, we can eliminate the element w_6 :

$$w_6 = w_2 - w_3 - w_4 + 2w_5.$$

Notice that

$$w_6 \in T_3 \circ T_3$$

and

$$w_1 \in (((T_2 \circ T_1) \circ T_1) \circ T_1) \circ T_1$$

where $((T_2 \circ T_1) \circ T_1) \circ T_1 \in T_5$. That is,

$$w_1 \in T_5 \circ T_1.$$

The same holds for w_2 and w_3 ,

$$w_2 \in T_5 \circ T_1, \quad w_3 \in T_5 \circ T_1.$$

And for w_4 and w_5 we define

$$w_4 \in T_4 \circ T_2, \quad w_5 \in T_4 \circ T_2.$$

From the identity $w_6 = w_2 - w_3 - w_4 + 2w_5$, it follows that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

That is,

$$T_6 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

D. Degree $n = 7$

For degree $n = 7$, we have 11 elements:

$$\begin{aligned} w_1 &= (((((x \circ x) \circ x) \circ x) \circ x) \circ x) \circ x, & w_2 &= (((((x \circ x) \circ (x \circ x)) \circ x) \circ x) \circ x) \circ x, \\ w_3 &= (((((x \circ x) \circ x) \circ (x \circ x)) \circ x) \circ x) \circ x, & w_4 &= (((((x \circ x) \circ x) \circ x) \circ (x \circ x)) \circ x) \circ x, \\ w_5 &= (((((x \circ x) \circ (x \circ x)) \circ (x \circ x)) \circ x) \circ x) \circ x, & w_6 &= (((((x \circ x) \circ x) \circ (x \circ x) \circ x)) \circ x) \circ x, \\ w_7 &= (((((x \circ x) \circ x) \circ x) \circ x) \circ (x \circ x)) \circ x) \circ x, & w_8 &= (((((x \circ x) \circ (x \circ x)) \circ x) \circ (x \circ x)) \circ x) \circ x, \\ w_9 &= (((((x \circ x) \circ x) \circ (x \circ x)) \circ (x \circ x)) \circ x) \circ x, & w_{10} &= (((((x \circ x) \circ x) \circ x) \circ (x \circ x) \circ x)) \circ x, \\ w_{11} &= (((((x \circ x) \circ (x \circ x)) \circ (x \circ x) \circ x)) \circ x) \circ x. \end{aligned}$$

Since $p(7-2) = p(5) = 7$, these 11 elements cannot be linearly independent. Thus, four linear relations occur among these eleven candidate elements.: We eliminate w_6 since in the $n = 6$ case we concluded that

$$T_3 \circ T_3 \subseteq T_5 \circ T_1 + T_4 \circ T_2.$$

For our calculations, we will use *Wolfram Mathematica* (see Appendix C). By this, we obtained

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \lambda_{10} + 2\lambda_{11}, \\ \lambda_4 &= -7\lambda_{10} - 12\lambda_{11} - 3\lambda_3, \\ \lambda_5 &= 6\lambda_{10} + 10\lambda_{11} + 2\lambda_3, \\ \lambda_7 &= 5\lambda_{10} + 8\lambda_{11} + 2\lambda_3, \\ \lambda_8 &= -6\lambda_{10} - 9\lambda_{11} - 3\lambda_3, \\ \lambda_9 &= \lambda_3. \end{aligned}$$

Taking

$$(\lambda_3, \lambda_{10}, \lambda_{11}) = (0, 1, 0), \quad (0, 0, 1), \quad (1, 0, 0),$$

we obtain the following three identities:

$$w_{10} + w_2 - 7w_4 + 6w_5 + 5w_7 - 6w_8 = 0, \quad (6a)$$

$$w_{11} + 2w_2 - 12w_4 + 10w_5 + 8w_7 - 9w_8 = 0, \quad (6b)$$

$$w_3 - 3w_4 + 2w_5 + 2w_7 - 3w_8 + w_9 = 0. \quad (6c)$$

The third relation (6c) coincides with the jetken polynomial, which is obtained from the Besken identity by taking $a = x$ and $b = x \circ x$ [1]. Identity (6a) allows us to eliminate w_{10} , while identity (6b) allows us to eliminate w_{11} . The third identity (6c) eliminates one more element from the remaining set. Thus, we obtain 7 linearly independent elements.

Since

$$w_{10}, w_{11} \in T_4 \circ T_3, \quad w_2, w_4, w_5 \in T_6 \circ T_1, \quad w_7, w_8 \in T_5 \circ T_2,$$

and

$$w_{10} = -w_2 + 7w_4 - 6w_5 - 5w_7 + 6w_8, \quad w_{11} = -2w_2 + 12w_4 - 10w_5 - 8w_7 + 9w_8,$$

it follows that

$$T_4 \circ T_3 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

That is,

$$T_7 \subseteq T_6 \circ T_1 + T_5 \circ T_2.$$

□

These computations motivate the following conjecture below. Our computations show that, up to degree 7, the stronger recursive inclusion holds.

Conjecture II.3. Let T_n be the homogeneous component of degree n in the one-generated case. Then,

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \quad n \geq 3.$$

These observations may be helpful in the study of the problem at higher degrees.

III. RESULTS AND DISCUSSION

In this work, we studied the homogeneous components T_n in the one-generated case. Our first main result gives the exact dimensions of these components:

$$\dim T_1 = \dim T_2 = 1, \quad \dim T_n = p(n-2) \quad \text{for all } n \geq 3.$$

Thus, the dimensions of the homogeneous components are described in combinatorial terms by the partition numbers.

We next describe the first homogeneous components in the $\circ$ -language and analyze the relations among their candidate basis elements. In degrees 1 through 5, the elements under consideration are linearly independent. The first nontrivial relation appears in degree 6. More precisely, by passing from the $\circ$ -language to the differential-monomial realization of the free Novikov algebra, we obtain identity (5), which is a consequence of the Tortken identity. This shows that in degree 6 the list of candidate elements is no longer independent, and one of them can be expressed through the others.

The same persists in degree 7, where further relations arise among the candidate elements. Together with the identity obtained in degree 6, these relations refine the structure of the homogeneous component T_7 . Thus, beginning with degree 6, the description of T_n requires not only the enumeration of all possible $\circ$ -elements, but also a careful analysis of the identities they satisfy.

Another main result is the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \quad n \leq 7.$$

This inclusion gives a relation between consecutive homogeneous components and shows that, in low degrees, elements of T_n can be obtained from elements of smaller degree by means of the symmetrized product.

A. Discussion

The obtained results show that even the one-generated case is already nontrivial. Although the dimensions of the homogeneous components admit a simple description, the identities appearing in degrees 6 and 7 show that the space of symmetric elements cannot be understood by listing $\circ$ -elements alone. One must also take into account the relations between them.

These observations indicate that the one-generated case already reflects essential features of Tortken elements. In particular, the low-degree computations reveal a regular pattern in the structure of the homogeneous components, which may remain visible in higher degrees.

The results obtained here may therefore be useful for the further study of Tortken elements in higher degrees and for the investigation of the case of several generators.

IV. CONCLUSION AND FUTURE WORK

In this paper, we studied the one-generated case of the space of Tortken elements in a free Novikov algebra. We proved the dimension formula

$$\dim T_1 = \dim T_2 = 1, \quad \dim T_n = p(n-2) \quad \text{for all } n \geq 3,$$

described the first homogeneous components in the $\circ$ -language, obtained explicit identities in degrees 6 and 7, and established the recursive inclusion

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \quad n \leq 7.$$

These results provide a precise description of the low-degree structure of the one-generated case. In particular, they indicate that the one-generated case already captures essential features of the structure of special Tortken algebras.

A natural problem for future work is to determine whether the recursive inclusion extends to all degrees. This leads to the conjecture

$$T_n \subseteq T_{n-1} \circ T_1 + T_{n-2} \circ T_2, \quad n \geq 3.$$

If true, this conjecture would provide a systematic way to describe the higher-degree components in the one-generated case. It may also serve as a useful starting point for studying the operadic and structural properties of special Tortken algebras and their connections with related Novikov-type algebras, such as δ -Novikov algebras, Gelfand–Dorfman algebras, and Novikov–Poisson algebras.

V. ACKNOWLEDGMENT

APPENDIX

A. Implementation of the derivation and the symmetrized product

The derivation was implemented by the following rules. The symmetrized product $a \circ b = (ab)'$ was implemented as $t[a, b]$.

```
Dr[C_Integer a_] := C Dr[a]

Dr[a_ + b_] := Dr[a] + Dr[b]

Dr[C_Integer a_ + K_Integer b_] := C Dr[a] + K Dr[b]

Dr[C_Integer a_ + K_Integer b_ + D_Integer d_] :=
  C Dr[a] + K Dr[b] + D Dr[d]

Dr[a_ b_] := Dr[a] b + a Dr[b]

Dr[a_^n_Integer] := Dr[a^(n - 1)] a + a^(n - 1) Dr[a]

Dr'[x_] := 0

Dr[0] := 0

t[a_, b_] := ExpandAll[Dr[a b]]
```

B. Degree 6 computation

Here we present the computation in degree 6. The following six candidate elements of degree six in the $\circ$ -language were used.

```
Genldeg6 = {
  gld6[1] = t[t[t[t[x, x], x], x], x], x],
  gld6[2] = t[t[t[x, x], t[x, x]], t[x, x]],
  gld6[3] = t[t[t[x, x], x], t[t[x, x], x]],
  gld6[4] = t[t[t[t[x, x], x], x], t[x, x]],
  gld6[5] = t[t[t[t[x, x], t[x, x]], x], x],
  gld6[6] = t[t[t[t[x, x], x], t[x, x]], x]
};
```

The program then determines the relations among the coefficients λ_i and computes their admissible values.

```
Combinldeg6 =
  Coefficient[
```

```

Collect[
  ExpandAll[Sum[Subscript[lambda, i] gld6[i], {i, 1, 6}]],
  {
    x Dr[x]^5,
    x^2 Dr[x]^3 Dr[Dr[x]],
    x^3 Dr[x] Dr[Dr[x]]^2,
    x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
    x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
    x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
    x^5 Dr[Dr[Dr[Dr[Dr[x]]]]
  }
],
{
  x Dr[x]^5,
  x^2 Dr[x]^3 Dr[Dr[x]],
  x^3 Dr[x] Dr[Dr[x]]^2,
  x^3 Dr[x]^2 Dr[Dr[Dr[x]]],
  x^4 Dr[Dr[x]]*Dr[Dr[Dr[x]]],
  x^4 Dr[x]*Dr[Dr[Dr[Dr[x]]]],
  x^5 Dr[Dr[Dr[Dr[Dr[x]]]]
}
];

S[x_] := x == 0;

TableForm[
  First[
    Solve[
      Map[S, Combin1deg6],
      Array[Subscript[lambda, #] &, 6]
    ]
  ]
]

```

C. Degree 7 computation

Here we present the computation in degree 7.

```

Gen1deg7 = {
  gld7[1] = t[t[t[x, x], x], x], t[t[x, x], x]],
  gld7[2] = t[t[t[x, x], t[x, x]], t[t[x, x], x]],
  gld7[3] = t[t[t[t[t[x, x], x], x], x], x], x],
  gld7[4] = t[t[t[t[x, x], t[x, x]], x], x], x],
  gld7[5] = t[t[t[t[x, x], x], t[x, x]], x], x],
  gld7[6] = t[t[t[t[x, x], x], x], t[x, x]], x],
  gld7[7] = t[t[t[x, x], t[x, x]], t[x, x]], x],
  gld7[8] = t[t[t[t[x, x], x], x], x], t[x, x]],
  gld7[9] = t[t[t[x, x], t[x, x]], x], t[x, x]],
  gld7[10] = t[t[t[x, x], x], t[x, x]], t[x, x]]
};

Combin1deg7 =

```

```

Coefficient[
Collect[
ExpandAll[Sum[Subscript[\[Lambda], i] g1d7[i], {i, 1, 10}]],
{
x Dr[x]^6,
x^2 Dr[x]^4 Dr[Dr[x]],
x^3 Dr[x]^2 Dr[Dr[x]]^2,
x^4 Dr[Dr[x]]^3,
x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
x^5 Dr[Dr[Dr[x]]]^2,
x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
}
],
{
x Dr[x]^6,
x^2 Dr[x]^4 Dr[Dr[x]],
x^3 Dr[x]^2 Dr[Dr[x]]^2,
x^4 Dr[Dr[x]]^3,
x^3 Dr[x]^3 Dr[Dr[Dr[x]]],
x^4 Dr[x] Dr[Dr[x]] Dr[Dr[Dr[x]]],
x^5 Dr[Dr[Dr[x]]]^2,
x^4 Dr[x]^2 Dr[Dr[Dr[Dr[x]]]],
x^5 Dr[Dr[x]] Dr[Dr[Dr[Dr[x]]]],
x^5 Dr[x] Dr[Dr[Dr[Dr[Dr[x]]]]],
x^6 Dr[Dr[Dr[Dr[Dr[Dr[x]]]]]]
}
];

S[x_] := x == 0;

TableForm[
First[
Solve[
Map[S, Combin1deg7],
Array[Subscript[\[Lambda], #] &, 10]
]
]
]

```

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